The Real Business Cycle Model Part 2

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Macroeconomics II

Introducing Labor

Introducing labor

- So far, changes in the capital stock are the only amplification mechanism.
- We have seen this is not sufficient, i.e., output is not volatile enough relative to the data.
- Endogenizing labor might lead to more amplification.
  - Labor is another factor of production that is procyclical.
  - Labor can adjust instantaneously.
  - Labor is more important in production than capital.
- We use again a model where production takes place at the household level.

The utility function

$$U_t = \frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \tag{1}$$

- Working causes disutility.
- ϕ scales the disutility of work relative to consumption.
- η is a measure related to the labor supply elasticity (more on this below).

The household problem

$$\max_{C_t, K_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{2}$$

s.t.

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t \tag{3}$$

$$Y_t = K_t^{\alpha} (A_t H_t)^{1-\alpha} \tag{4}$$

$$I_t = K_{t+1} - (1-\delta)K_t \tag{5}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \tag{6}$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0 \Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) - \lambda_t [C_t + K_{t+1} - K_t^{\alpha} (A_t H_t)^{1-\alpha} - (1-\delta) K_t] \Big] \Big\}. \quad (7)$$

$$\frac{\partial \Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \quad (8)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(\alpha K_{t+1}^{\alpha-1} (A_{t+1} H_{t+1})^{1-\alpha} + (1-\delta) \right) \Big\} \quad (9)$$

$$\frac{\partial \Lambda_t}{\partial H_t} : \phi H_t^{\eta} = \lambda_t (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha} \quad (10)$$

Intertemporal optimality

Combining the optimality conditions yields the Euler equation:

$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(\alpha K_{t+1}^{\alpha-1} \left(A_{t+1} H_{t+1} \right)^{1-\alpha} + (1-\delta) \right) \Big\} \qquad (11)$$

The gain of consuming today (the marginal utility of consumption) = the expected gain from deferring consumption (the expectations over marginal utility of consumption tomorrow times the return on savings).

Intratemporal optimality

Optimal labor supply:

$$\phi H_t^{\eta} = C_t^{-\gamma}(1-\alpha)K_t^{\alpha}A_t^{1-\alpha}H_t^{-\alpha} \quad (12)$$

The marginal disutility of working = the marginal gain from working (marginal utility of consumption times the marginal product of labor.

Note, this is a within-period decision. Think about an increase in productivity today:

- Increases MPL ⇒ increases hours worked (substitution effect).
- Decreases MUC ⇒ decreases hours worked (wealth effect).

Understanding η

Before turning to the model solution, let us briefly discuss the interpretation of η. The labor supply elasticity is given by:

$$\frac{\partial H_t}{\partial MPL_t}\frac{MPL_t}{H_t},$$

with

$$MPL_t = (1-\alpha)K_t^{\alpha}A_t^{1-\alpha}H_t^{-\alpha}. \tag{13}$$

Understanding η II

Assume there is no wealth effect, $\frac{\partial C_t}{\partial MPL_t} = 0$:

$$\frac{\partial H_t}{\partial MPL_t}\frac{MPL_t}{H_t} = \frac{1}{\eta}\left[\frac{1}{\phi}C_t^{-\gamma}\right]^{\frac{1}{\eta}} MPL_t^{\frac{1}{\eta}-1}\frac{MPL_t}{H_t}$$
$$= \frac{1}{\eta}\left[\frac{1}{\phi}C_t^{-\gamma}MPL_t\right]^{\frac{1}{\eta}}\frac{1}{H_t}$$
$$= \frac{1}{\eta}.$$

That is, $1/\eta$ is the elasticity of labor supply holding the marginal utility of consumption (the wealth effect) constant. This is called the Frisch elasticity.

Equilibrium

An equilibrium is a set of allocations (C_t, K_{t+1}, and H_t) taking K_t, A_t, and the stochastic process for A_t as given such that the budget constrained, (3), and the optimality conditions (11) and (12) hold.

The solution to the model is given by the following set of equations

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \tag{14}$$

$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(\alpha K_{t+1}^{\alpha-1} \left(A_{t+1} H_{t+1} \right)^{1-\alpha} + (1-\delta) \right) \Big\} \tag{15}$$

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t \tag{16}$$

$$Y_t = \left(A_t H_t \right)^{1-\alpha} K_t^{\alpha} \tag{17}$$

$$I_t = K_{t+1} - (1-\delta) K_t \tag{18}$$

$$\phi H_t^{\eta} = C_t^{-\gamma} (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha} \tag{19}$$

$$\tag{20}$$

Solving for the steady state

Let us set again $\epsilon_t = 0$ and postulate $K_{t+1} = K_t$ and $C_{t+1} = C_t$. Moreover, define $k^{ss} = \frac{K^{ss}}{H^{ss}}$:

$$\frac{K^{ss}}{H^{ss}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}} \tag{21}$$

$$\frac{C^{ss}}{H^{ss}} = (k^{ss})^{\alpha} - \delta k^{ss} \tag{22}$$

$$\frac{I^{ss}}{H^{ss}} = \delta k^{ss} \tag{23}$$

$$\frac{Y^{ss}}{H^{ss}} = (k^{ss})^{\alpha} \tag{24}$$

$$H^{ss} = \left[\frac{1}{\phi} \frac{(1-\alpha)(k^{ss})^{\alpha}}{[(k^{ss})^{\alpha} - \delta k^{ss}]^{\gamma}} \right]^{\frac{1}{\eta+\gamma}} . \tag{25}$$

Log linearizing output

We are going to study again a first-order approximation around the steady state. Starting with the production function:

$$Y_t = (A_t H_t)^{1-\alpha} K_t^{\alpha} \tag{26}$$

Log-linearization

$$Y^{ss}(1+\hat{Y}_t) = (A^{ss})^{1-\alpha}(H^{ss})^{1-\alpha}(K^{ss})^{\alpha}(1+(1-\alpha)\hat{A}_t+(1-\alpha)\hat{H}_t+\alpha\hat{K}_t)$$

$$\hat{Y}_t = (1-\alpha)\hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{H}_t. \tag{27}$$

Hence, if H_t is procyclical, output will be more volatile than productivity.

Log linearizing Euler equation

$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(\alpha K_{t+1}^{\alpha-1} \left(A_{t+1} H_{t+1} \right)^{1-\alpha} + (1-\delta) \right) \Big\} \tag{28}$$

Log-linearization:

$$\begin{aligned} C^{ss}(1-\gamma\hat{C}_t) = \beta\mathbb{E}_t \Big\{ & C^{ss}(1-\gamma\hat{C}_{t+1}) \Big[1-\delta+\alpha\left(K^{ss}\right)^{\alpha-1}\left(H^{ss}\right)^{1-\alpha} \\ & \left(1+(1-\alpha)\hat{A}_{t+1}+(1-\alpha)\hat{H}_{t+1}+(\alpha-1)\hat{K}_{t+1} \right) \Big] \Big\} \end{aligned} \tag{29}$$

$$\begin{aligned} \mathbb{E}_t\hat{C}_{t+1}-\hat{C}_t = & \frac{1}{\gamma}(1-\beta(1-\delta)) \\ & \mathbb{E}_t\Big[(1-\alpha)\hat{A}_{t+1}+(1-\alpha)\hat{H}_{t+1}+(\alpha-1)\hat{K}_{t+1} \Big]. \end{aligned} \tag{30}$$

Log linearizing Euler equation II

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1-\beta(1-\delta)) \mathbb{E}_t \Big[(1-\alpha)\hat{A}_{t+1} + (1-\alpha)\hat{H}_{t+1} + (\alpha-1)\hat{K}_{t+1}\Big]. \quad (31)$$

Consumption growth is high when:

- expected productivity is high tomorrow.
- expected hours worked are high tomorrow.
- expected capital is low tomorrow.

Log linearizing hours worked

Optimal hours worked:

$$\phi H_t^{\eta+\alpha} = C_t^{-\gamma}(1-\alpha)K_t^{\alpha}A_t^{1-\alpha} \tag{32}$$

Log-linearization

$$(H^{ss})^{\eta+\alpha}\phi(1+(\eta+\alpha)\hat{H}_t) = \\ (1-\alpha)(C^{ss})^{-\gamma}(K^{ss})^{\alpha}(1-\gamma\hat{C}_t+\alpha\hat{K}_t+(1-\alpha)\hat{A}_t) \tag{33}$$

$$\hat{H}_t = \frac{1}{\eta+\alpha}[-\gamma\hat{C}_t+\alpha\hat{K}_t+(1-\alpha)\hat{A}_t]. \tag{34}$$

The cyclical movement of hours worked

$$\hat{H}_t = \frac{1}{\eta+\alpha}[-\gamma\hat{C}_t + \alpha\hat{K}_t + (1-\alpha)\hat{A}_t]. \tag{35}$$

- After an increase in productivity, hours increase ($\hat{A}_t$, substitution effect) or decrease ($\hat{C}_t$, wealth effect) .

The cyclical movement of hours worked

$$\hat{H}_t = \frac{1}{\eta+\alpha}[-\gamma\hat{C}_t + \alpha\hat{K}_t + (1-\alpha)\hat{A}_t]. \tag{35}$$

- The ensuring capital accumulation increases hours ($\hat{K}_t$) but the growing consumption ($\hat{C}_t$) depresses them.

$$\hat{H}_t = \frac{1}{\eta+\alpha}[-\gamma \hat{C}_t + \alpha \hat{K}_t + (1-\alpha)\hat{A}_t]. \tag{35}$$

- Responses are stronger when the labor supply elasticity is larger.

Amplification mechanism of hours worked

$$\hat{H}_t = \frac{1}{\eta + \alpha}[-\gamma \hat{C}_t + \alpha \hat{K}_t + (1-\alpha)\hat{A}_t]. \quad (36)$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1 - \beta(1-\delta)) \mathbb{E}_t \Big[(1-\alpha)\hat{A}_{t+1} + (1-\alpha)\hat{H}_{t+1} + (\alpha - 1)\hat{K}_{t+1}\Big]. \quad (37)$$

Consider a positive shock ϵ_t. As a result, $\hat{A}_{t+1} > 0$:

- Consumption growth is positive, i.e., households defer consumption and increase investment.
- Consumption today is lower than in a static model. This allows the substitution effect to dominate the wealth effect, i.e., $\hat{H}_t > 0$.
- Rising hours raise the MPK. This leads to yet more willingness to defer consumption.

Calibration

- Using changes in tax policies, the micro literature finds elasticities of hours with respect to wages of around 0.5 implying $\eta = 2$.
- Commonly, ϕ is calibrated such that households spend $1/3$ of their time working in steady state: $\phi = 30$.
- Remember, in the data, we measure

$$\ln A_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln H_t \tag{38}$$

Yet, our model assumes

$$(1-\alpha) \ln A_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln H_t \tag{39}$$

Given a target for the standard deviation of TFP of 1.25% and our $AR(1)$ process, we require: $Std. \ln A_t = \frac{1.25\%}{1-\alpha}$.

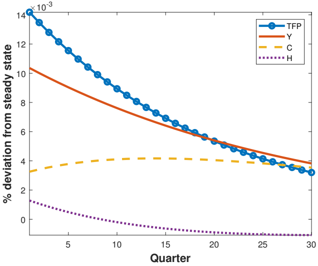
Comparing model and data

	Data				
	Y	C	I	H	TFP
Std. %	1.61	1.25	7.27	1.9	1.25
ACR(1)	0.78	0.68	0.78	0.81	0.76
	Model				
Std. %	1.35	0.45	4.37	0.18	1.24
ACR(1)	0.72	0.76	0.71	0.73	0.71

Comparing model and data

	Y	C	I	H	TFP
			Data		
Y	1				
C	0.78	1			
I	0.83	0.67	1		
H	0.87	0.68	0.76	1	
TFP	0.79	0.71	0.77	0.49	1
			Model		
Y	1				
C	0.96	1			
I	1	0.93	1		
H	0.89	0.73	0.93	1	
TFP	1	0.94	1	0.92	1

Understanding cyclicality of hours



- Hours increase after an increase in TFP but by less than $1/\eta$ because consumption increases.
- Over time:
  - *MPL* returns to its origin.
  - Consumption stays relatively high because of high K.
  - The hours response turns negative.
- The weak hours response gives little additional amplification.

Comparing model with data

The successes:

- Hours co-move positively with other macroeconomic aggregates.
- The model has a little bit more propagation.
- The co-movement with TFP is weak relative to other aggregates.

  Wealth effects are important late in the cycle.

The misses:

- Hours are not nearly as volatile enough.
- The co-movement between hours and macroeconomic aggregates is yet too strong.

The volatility of hours

$$\hat{H}_t = \frac{1}{\eta + \alpha}[-\gamma \hat{C}_t + \alpha \hat{K}_t + (1 - \alpha)\hat{A}_t]. \tag{40}$$

- The volatility is increasing in the labor supply elasticity.
- Can we rationalize a volatility that is higher than that of micro studies?
- We will see later that most hours adjustment over the business cycle is at the extensive margin. This is not the focus of the micro studies.
- We now use a calibration with $\eta = 0.5$ which implies $\phi = 6$.

Comparing model and data

	Data				
	Y	C	I	H	TFP
Std. %	1.61	1.25	7.27	1.9	1.25
	Model $\eta = 2$				
Std. %	1.35	0.45	4.37	0.18	1.24
	Model $\eta = 0.5$				
Std. %	1.56	0.45	5.30	0.52	1.24

Comparing model with data

- A higher labor supply elasticity helps to increase the volatility of hours.
- In general, it leads to more amplification:
  - Volatile hours lead to volatile MPK and, hence, investment.
  - Volatile hours and investment make output more volatile.

Back to the recursive formulation

Before finishing this topic, we study again how to solve the model globally. The goal is to find policy functions $\mathbf{C}(K, A), \mathbf{H}(K, A)$ that solve:

$$V(K,A) = \max_{C,K',H} \left\{ \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) + \beta \mathbb{E} V(K',A') \right\}$$

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t$$

$$Y_t = K_t^{\alpha} (A_t H_t)^{1-\alpha}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1}$$

Back to the recursive formulation

We have seen that the first order condition is:

$$H_t = \left[\frac{1}{\phi} C_t^{-\gamma} (1-\alpha) K_t^{\alpha} A_t^{1-\alpha}\right]^{\frac{1}{\eta+\alpha}}.$$

Back to the recursive formulation

We have seen that the first order condition is:

$$H_t = \left[\frac{1}{\phi} C_t^{-\gamma} (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} \right]^{\frac{1}{\eta+\alpha}} .$$

Hence, knowing optimal policy $\mathbf{C}(K, A)$ allows us to compute $\mathbf{H}(K, A)$.

A solution algorithm

- One way to solve the problem is:
  1. Guess optimal policy for labor, $\mathbf{H}(K, A)$.
  2. Solve for optimal policy for consumption $\mathbf{C}(K, A)$.
  3. Solve FOC for optimal $\mathbf{H}(K, A)$.
  4. Update $\mathbf{H}(K, A)$.
  5. Iterate until convergence.

Decentralizing the Economy

Decentralizing the economy

- So far, all production took place at the household level.
- As a result, we do not have any prices.
- We are now introducing firms and, hence, also derive the cyclical behavior of prices.
- We assume households own the factors of production and rent them to firms.
- Households own the firms and firms distribute their profits, Π_t, to the households.
- Firms operate in perfectly competitive factor and product markets.

The household problem

Let W_t be the real wage and R_t be the real rental price of capital. The household takes these prices as given:

$$\max_{C_t, K_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{41}$$

s.t.

$$C_t + K_{t+1} = W_t H_t + R_t K_t + \Pi_t + (1-\delta) K_t \tag{42}$$

$$I_t = K_{t+1} - (1-\delta) K_t \tag{43}$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0 \Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) - \lambda_t [C_t + K_{t+1} - W_t H_t - R_t K_t - \Pi_t - (1-\delta) K_t] \Big] \Big\}. \quad (44)$$

$$\frac{\partial \Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \quad (45)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(R_{t+1} + (1-\delta) \right) \Big\} \quad (46)$$

$$\frac{\partial \Lambda_t}{\partial H_t} : \phi H_t^{\eta} = \lambda_t W_t \quad (47)$$

Intertemporal optimality

Euler equation:

$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(R_{t+1} + (1 - \delta) \right) \Big\} \tag{48}$$

The gain of consuming today (the marginal utility of consumption) = the gain from deferring consumption (the expectation over the marginal utility of consumption tomorrow times the returns on savings).

Intratemporal optimality

Optimal labor supply:

$$\phi H_t^{\eta} = C_t^{-\gamma} W_t \tag{49}$$

The marginal disutility of working = the marginal gain from working

(marginal utility of consumption times the wage rate).

The firm problem

- Firms are owned by the households.
- They maximize discounted dividends that they pay to the households. Per period dividends are:

$$\Pi_t = K_t^{\alpha} \left(A_t H_t\right)^{1-\alpha} - W_t H_t - R_t K_t. \tag{50}$$

- They operate in perfectly competitive markets and, hence, take prices as given.

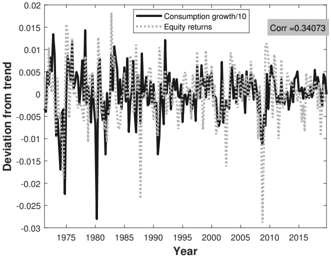
The stochastic discount factor

As firms are owned by the households, they discount future profits the same way the household does. Note:

$$1 = \mathbb{E}_t \left\{ \frac{\beta C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(R_{t+1} + (1 - \delta) \right) \right\} \quad (51)$$

- Households discount future utility flows with β.
- It would be tentative to think that they also discount future resource flows with β.
- However, this is not the case. They discount these with: $\frac{\beta \mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}$.
- This implies that discounting of future returns is higher when today's consumption is low.

Consumption growth and equity returns



- Finance tells us that stock prices are discounted future returns.
- I compute equity returns from the Wilshire 500 index.
- Indeed, in the data, consumption growth and equity returns are positively correlated.

The firm problem II

Hence, the firm discounts a unit of dividend in period t that generates $C_t^{-\gamma}$ units of utility back to the initial period ($t = 0$). We need to measure this utility stream relative to the value of paying the dividend in period $t = 0$:

$$\max_{K_t, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{-\gamma}}{C_0^{-\gamma}} \left[K_t^{\alpha} \left(A_t H_t \right)^{1-\alpha} - W_t H_t - R_t K_t \right] \right\} \tag{52}$$

and

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \tag{53}$$

First order conditions

$$R_t = \alpha K_t^{\alpha-1} (A_t H_t)^{1-\alpha} \tag{54}$$

$$W_t = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha} \tag{55}$$

- Factors own their marginal products. Hence, given the constant returns to scale production function, profits are zero.

Equivalence of the centralized and decentralized economy

Plug the price of capital into the Euler equation (50) to obtain:

$$C_t^{-\gamma} = \mathbb{E}_t \Big\{ \beta C_{t+1}^{-\gamma} \left(\alpha K_{t+1}^{\alpha-1} (A_{t+1} H_{t+1})^{1-\alpha} + (1-\delta) \right) \Big\}, \tag{56}$$

which is the Euler equation from the centralized economy. Moreover, use $\Pi_t = 0$ and prices in the household's budget constraint which brings us back to the national income identity:

$$Y_t = C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{I_t}. \tag{57}$$

As you have seen in Macro I, the equivalence of the two economies is a consequence of the welfare theorems holding.

Equilibrium

A competitive equilibrium is a set of allocations (C_t, K_{t+1}, and H_t) and prices (R_t and W_t) taking K_t, A_t, and the stochastic process for A_t as given such that the budget constrained, (41), the optimality conditions (50) and (48), and the demand for capital, (53), and labor, (54), hold.

Solving for the steady state

$$R^{ss} = \frac{1}{\beta} - 1 + \delta \quad (58)$$

$$\frac{K^{ss}}{H^{ss}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}} \quad (59)$$

$$\frac{C^{ss}}{H^{ss}} = (k^{ss})^{\alpha} - \delta k^{ss} \quad (60)$$

$$\frac{I^{ss}}{H^{ss}} = \delta k^{ss} \quad (61)$$

$$\frac{Y^{ss}}{H^{ss}} = (k^{ss})^{\alpha} \quad (62)$$

$$W^{ss} = (1-\alpha)(k^{ss})^{\alpha} \quad (63)$$

$$H^{ss} = \left[\frac{1}{\phi}\frac{(1-\alpha)(k^{ss})^{\alpha}}{[(k^{ss})^{\alpha} - \delta k^{ss}]^{\gamma}}\right]^{\frac{1}{\eta+\gamma}}. \quad (64)$$

Understanding factor price movements

$$R_t = \alpha K_t^{\alpha-1} (A_t H_t)^{1-\alpha} \quad (65)$$
$$W_t = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha} \quad (66)$$

Log-linearize using **LI Rule 1** and **LI Rule 4** yields:

$$R^{ss}(1+\hat{R}_t) = \alpha (k^{ss})^{\alpha-1}(1+(\alpha-1)\hat{k}_t + (1-\alpha)\hat{A}_t) \quad (67)$$
$$\hat{R}_t = (\alpha-1)\hat{k}_t + (1-\alpha)\hat{A}_t \quad (68)$$

$$W^{ss}(1+\hat{W}_t) = (1-\alpha)(k^{ss})^{\alpha}(1+\alpha\hat{k}_t + (1-\alpha)\hat{A}_t) \quad (69)$$
$$\hat{W}_t = \alpha\hat{k}_t + (1-\alpha)\hat{A}_t \quad (70)$$

Understanding factor price movements II

$$\hat{R}_t = (\alpha - 1)\hat{k}_t + (1 - \alpha)\hat{A}_t$$
$$\hat{W}_t = \alpha\hat{k}_t + (1 - \alpha)\hat{A}_t$$

- After an increase in productivity, both factor prices increase by approximately the same amount.

Understanding factor price movements II

$$\hat{R}_t = (\alpha - 1)\hat{k}_t + (1-\alpha)\hat{A}_t$$
$$\hat{W}_t = \alpha\hat{k}_t + (1-\alpha)\hat{A}_t$$

- Over time, as the economy accumulates capital, wages stay higher relative to the interest rate.

The cyclical movement of hours and wages

$$\phi H_t^\eta = C_t^{-\gamma} W_t \tag{71}$$

Log-linearize using **LI Rule 1** and **LI Rule 4** yields

$$(H^{ss})^\eta (1 + \eta \hat{H}_t) = (C^{ss})^{-\gamma} W^{ss} \frac{1}{\phi} (1 - \gamma \hat{C}_t + \hat{W}) \tag{72}$$

$$\hat{H}_t = \frac{1}{\eta} [-\gamma \hat{C}_t + \hat{W}_t]. \tag{73}$$

- We have a proportional relationship between wages and hours worked.
- The strength of the response depends on the labor supply elasticity.
- In the data, hours are twice as volatile as wages. Hence, we need $\eta < 0.5$.

Results I

	Y	C	I	H	TFP	w	r
				Data			
Std. %	1.61	1.25	7.27	1.9	1.25	0.96	1.02
ACR(1)	0.78	0.68	0.78	0.81	0.76	0.66	0.71
				$\eta = 2$			
Std. %	1.35	0.45	4.37	0.18	1.24	1.19	0.05
ACR(1)	0.72	0.76	0.71	0.73	0.71	0.73	0.71
				$\eta = 0.001$			
Std. %	1.99	0.48	7.09	1.19	1.24	0.96	0.07
ACR(1)	1.71	0.78	0.7	0.71	0.71	0.78	0.7

Results II

	Y	C	I	H	TFP	w	r
				Data			
Y	1						
C	0.78	1					
I	0.83	0.67	1				
H	0.87	0.68	0.76	1			
TFP	0.79	0.71	0.77	0.49	1		
w	0.12	0.29	0.07	-0.06	0.34	1	
r	0.24	0.11	0.20	0.40	0.05	-0.13	1
				$\eta = 0.001$			
Y	1						
C	0.91	1					
I	1	0.87	1				
H	0.94	0.72	0.97	1			
TFP	1	0.93	0.99	0.93	1		
w	0.91	1	0.87	0.72	0.93	1	
r	0.96	0.75	0.98	1	0.94	0.75	1

Taking stock

- The model implies the positive co-movement between hours and other aggregates.
- For hours to move sufficient over the cycle, we require huge labor supply elasticities because of a strong wealth effect.
- Introducing hours creates little amplification. The reason is the strong wealth effect.
- The model predicts a strong co-movement between prices and quantities. This is absent in the data.
- Interest rates are not nearly volatile enough.

The weak link between prices and quantities

- We have a single shock model. Maybe we require additional shocks.

  We will consider interest rate shocks later in the course.

- Hours worked may be a poor measure of labor quantity.

  In a recession, low-educated lose their job first.

  Implies that aggregate wages may move little.

Asset pricing

Asset pricing

- We have seen how to compute the price of capital (equity).
- We will now compare the price of capital to the price of a safe asset (bonds, B_t).
- The bonds are provided by the firms and pay a certain return.
- The hope is that the RBC model explains the equity premium, i.e., the fact that stocks pay an average excess return over bonds of 6% annually.

The household problem

Households now choose how many bonds to accumulate. Note, the interest rate on bonds is pre-determined. The household knows today the return it will get tomorrow:

$$\max_{C_t, K_{t+1}, B_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{74}$$

s.t.

$$C_t + K_{t+1} + B_{t+1} = W_t H_t + R_t K_t + \Pi_t + (1-\delta) K_t + (1 + r_{t-1}) B_t \tag{75}$$

$$I_t = K_{t+1} - (1-\delta) K_t \tag{76}$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0\Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right)$$
$$-\lambda_t [C_t + K_{t+1} + B_{t+1} - W_t H_t - R_t K_t - \Pi_t - (1-\delta) K_t - (1 + r_{t-1}) B_t] \Big] \Big\}. \quad (77)$$

$$\frac{\partial \Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \quad (78)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(R_{t+1} + (1-\delta) \right) \Big\} \quad (79)$$

$$\frac{\partial \Lambda_t}{\partial H_t} : \phi H_t^{\eta} = \lambda_t W_t \quad (80)$$

$$\frac{\partial \Lambda_t}{\partial B_{t+1}} : \beta^t \lambda_t = \beta^{t+1} \mathbb{E}_t \lambda_{t+1} (1 + r_t) \quad (81)$$

Optimal decisions for bonds

The only new condition is the one for bonds:

$$C_t^{-\gamma} = \beta \mathbb{E}_t C_{t+1}^{-\gamma}(1 + r_t) \tag{82}$$

The household is guaranteed return $1 + r_t$ tomorrow. When buying a bond today, the household gives up the marginal utility of consumption today and gains the discounted expected marginal utility tomorrow times the return on holding the bond.

The firm problem

Firms can now also pay dividends by issuing new bonds. They have to pay back last period's bonds:

$$\max_{K_t, H_t, B_{t+1}} \mathbb{E}_0 \Bigg\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{-\gamma}}{C_0^{-\gamma}} \Big[K_t^{\alpha} (A_t H_t)^{1-\alpha} - W_t H_t - R_t K_t + B_{t+1} - (1 + r_{t-1}) B_t \Big] \Bigg\}. \quad (83)$$

First order condition of bonds:

$$C_t^{-\gamma} = \beta(1 + r_t)\mathbb{E}_t C_{t+1}^{-\gamma}. \quad (84)$$

Note that the first order condition of firms and the household are the same. Hence, any level of bonds is an equilibrium.

Equity and bond returns

- Our goal is to understand the excess returns of equity over bonds.
- I will first show that studying the steady state is not very insightful.
- Similarly, a first order Taylor series expansion around the steady state brings little insight.
- The reason is that aggregate risk matters for asset prices.

Steady state bonds and stocks

$$r^{ss} = \frac{1}{\beta} - 1 \tag{85}$$

$$R^{ss} = \frac{1}{\beta} - 1 + \delta. \tag{86}$$

In steady state, $r^{ss} = R^{ss} - \delta$. This should be no surprise. In steady state, there is no uncertainty and, hence, returns on equity are fully predictable which makes them a perfect substitute to bonds.

Log-linearizing bonds and stocks

We take again a first-order Taylor series expansion:

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1-\beta)\hat{r}_t \tag{87}$$

$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}(1-\beta(1-\delta))\mathbb{E}_t \hat{R}_{t+1}. \tag{88}$$

With $\delta = 0$, we have the exact same dynamics for equity and bonds. The reason is that with a first-order approximation, risk has no effects on the dynamics of the system.

Comparing bonds and stocks

Let us now go back to the non-linear solution:

$$1 = \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} (1 + r_t) \tag{89}$$

$$1 = \mathbb{E}_t \left\{ \beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(R_{t+1} + (1 - \delta) \right) \right\} \tag{90}$$

Note that r_t is pre-determined but R_{t+1} and C_{t+1} are not. Technically, the difference between the two equations is that the expectation operator in the bond equation is only over consumption tomorrow while in the equity equation it is about the *joint behavior* of consumption and equity returns tomorrow.

Comparing bonds and stocks II

$$1 = \mathbb{E}_t\Big\{\beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}(1 + R_{t+1} - \delta)\Big\} \tag{91}$$

Remember, for two stochastic variables, X, Y, we have $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y + COV(X, Y)$:

$$1 = \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} + \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\mathbb{E}_t(R_{t+1} - \delta) + COV\left(\frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, R_{t+1} - \delta\right), \tag{92}$$

which has an additional covariance term relative to the bond equation:

$$1 = \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} + \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} r_t \tag{93}$$

Comparing bonds and stocks III

Combining the two optimality conditions:

$$\beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} r_t = \beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \mathbb{E}_t(R_{t+1} - \delta) + COV\left(\frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, R_{t+1} - \delta\right) \quad (94)$$

$$\beta \frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left[\mathbb{E}_t(R_{t+1} - \delta) - r_t\right] = -COV\left(\frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, R_{t+1} - \delta\right), \quad (95)$$

where the left-hand side is the (weighted) equity premium. We have seen that the covariance term is negative. Whenever, $\frac{\mathbb{E}_t C_{t+1}^{-\gamma}}{C_t^{-\gamma}}$ is large, $R_{t+1} - \delta$ must be small. For example, at the end of a boom, $C_t > C_{t+1}$ and R_{t+1} is small. The model predicts an equity premium because investors need to be compensated to defer consumption when the marginal product of capital is high.

We can derive further insights by writing the equation in terms of consumption growth and making distributional assumptions. Define $x_{t+1} = \ln C_{t+1} - \ln C_t$, bond returns $R_t^B = 1 + r_t$ and equity returns $R_{t+1}^E = 1 + R_{t+1} - \delta$. Then the optimality conditions become

$$\beta \mathbb{E}_t \exp(-\gamma x_{t+1} + \ln R_t^B) = 1 \tag{96}$$

$$\beta \mathbb{E}_t \exp(-\gamma x_{t+1} + \ln R_{t+1}^E) = 1. \tag{97}$$

Now assume x_{t+1} and $x_{t+1} + \ln R_{t+1}^E$ are normally distributed. For a variable $y \sim N(\bar{y}, \sigma^2)$ we have that $Y = \exp(y)$ is log-normally distributed and

$$\mathbb{E} Y = \exp\left(\bar{y} + \frac{1}{2}\sigma^2\right). \tag{98}$$

Comparing bonds and stocks V

The optimality conditions become

$$\beta \exp\Big(-\gamma \bar{x}_{t+1} + \ln R_t^B + \frac{1}{2} Var(-\gamma x_{t+1}) \Big) = 1 \tag{99}$$

$$\beta \exp\Big(-\gamma \bar{x}_{t+1} + \mathbb{E}_t \ln R_{t+1}^E + \frac{1}{2} Var(-\gamma x_{t+1} + \ln R_{t+1}^E) \Big) = 1. \tag{100}$$

Equate the equations and take logs to get

$$\mathbb{E}_t \ln R^E_{t+1} - \ln R^B_t = \frac{1}{2} Var(-\gamma x_{t+1}) - \frac{1}{2} Var(-\gamma x_{t+1} + \ln R^E_{t+1}) \quad (101)$$

$$= -\frac{1}{2}\sigma^2_{\ln R^E_{t+1}} + \gamma COV(x_{t+1}, \ln R^E_{t+1}), \quad (102)$$

Finally, note that:

$$\mathbb{E}_t R^E_{t+1} = \exp\left(\mathbb{E}_t \ln R^E_{t+1} + \frac{1}{2}\sigma^2_{\ln R^E_{t+1}}\right) \quad (103)$$

$$\ln \mathbb{E}_t R^E_{t+1} = \mathbb{E}_t \ln R^E_{t+1} + \frac{1}{2}\sigma^2_{\ln R^E_{t+1}}. \quad (104)$$

Comparing bonds and stocks VII

$$\underbrace{\ln \mathbb{E}_t R^E_{t+1} - \ln R^B_t}_{\text{equity premium}} = \gamma Corr(x_{t+1}, \ln R^E_{t+1}) \sigma_{x_{t+1}} \sigma_{\ln R^E_{t+1}}. \quad (105)$$

- Note, investors need to be compensated for the unconditional volatility of stocks, $\sigma_{\ln R^E_{t+1}}$. This is as in your standard undergraduate finance text book.
- The key new insight is that they also need to be compensated for the co-movement of stock returns and consumption growth.
- The equity premium is increasing in risk aversion.
- This insight carries over to any comparison of returns. German government bonds pay particular low average interest rates. Some argue: Crisis lead to a flight to safety and high conditional returns on German bonds.

Comparing bonds and stocks VIII

$$\ln \mathbb{E}_t R^E_{t+1} - \ln R^B_t = \gamma Corr(x_{t+1}, \ln R^E_{t+1}) \sigma_{x_{t+1}} \sigma_{\ln R^E_{t+1}}. \tag{106}$$

In US data, $\sigma_{x_{t+1}} = 0.036$, $\sigma_{\ln R^E_{t+1}} = 0.167$, and $Corr(x_{t+1}, \ln R^E_{t+1}) = 0.4$.

- With $\gamma = 2$ we have an equity premium of 0.48%.
- To get to an equity premium of around 6% we require $\gamma = 25$.
- With such a high risk aversion, the model fails to generate meaningful consumption volatility.
- Question: If consumption fluctuates little over the cycle, why do households demand such high insurance payments?

Critiques of the RBC model

Three common critiques of the RBC model:

1. Model performance:
   - The model shows too little amplification of technology shocks.
   - The model has problems matching the volatility of hours.
   - The model implies strong co-movement of aggregates and prices.
   - The model implies only a small equity premium.

2. Simplicity:
   - There is only one shock, surely other shocks matter.
   - There are no frictions affecting the business cycle.

3. Implication: The Great Recession was the "Great Vacation".